

REAL OPEN BOOKS AND REAL CONTACT STRUCTURES

FERİT ÖZTÜRK AND NERMİN SALEPCİ

ABSTRACT. A real 3-manifold is a smooth 3-manifold together with an orientation preserving smooth involution, called a real structure. We prove that every real 3-manifold admits a real open book - an open book compatible with the real structure. The proof also shows that every real Heegaard decomposition of a real 3-manifold comes from a real open book. This is not true in general in the non-real setting. We observe that there are infinitely many closed oriented real 3-manifolds which cannot have a maximal real Heegaard decomposition. Furthermore, we show that there is a Giroux correspondence in the existence of a real structure; namely we prove that there is a one to one correspondence between the real contact structures on a 3-manifold up to equivariant contact isotopy and the real open books up to positive real stabilization. Finally, we construct a real 3-manifold with a real open book which is the canonical boundary of a Lefschetz fibration but which cannot be filled by any real Lefschetz fibration.

1. INTRODUCTION AND BASIC DEFINITIONS

A *real structure* on an oriented $2n$ -manifold (or equally $2n - 1$ -manifold) X with boundary (possibly empty) is defined as an involution c_X on X which is orientation preserving if n is even and orientation reversing if n is odd and the fixed point set of which is of dimension n , if it is not empty. Hence, if M is the oriented boundary of an oriented $2n$ -manifold X , a real structure on X restricts to a real structure on M . We call a manifold together with a real structure a *real manifold* and the fixed point set of the real structure the *real part*.

If a symplectic structure (or a contact structure) pulls back to minus itself under a real structure c , such structures are called c -real and those manifolds are called c -real symplectic (or c -real contact respectively) manifolds. Real algebraic varieties and links of real algebraic isolated singularities supplied with natural structures are examples for real symplectic and contact manifolds.

Every closed oriented 3-manifold admits an open book decomposition [1] and a positive contact structure [18]. Furthermore the work of E. Giroux points out that there is a one to one correspondence between open book decompositions up to positive stabilization and positive contact structures up to isotopy [10] (see e.g. [4] or [6] for a careful discussion).

However not every 3-manifold admits a real structure; in fact the ones which do not admit a real structure are in abundance [25]. Nevertheless, once a real 3-manifold is given we can make appropriate definitions similar to the ones above (see Sections 1.1-3) and show that there are analogous relations in the existence of a real structure.

A *real open book decomposition* is an open book with the real structure preserving the page structure and leaving exactly two pages invariant. It is known that

every real 3-manifold (M, c_M) admits a *characteristic Heegaard splitting* [20, Proposition 2.4]; i.e. there is a Heegaard splitting with a c_M -invariant Heegaard surface and the two handlebodies are sent to each other by c_M . In the present work we use the term *c_M -real Heegaard splitting (or decomposition)* instead of characteristic Heegaard splitting. Using the existence of real Heegaard splittings we show that every real 3-manifold admits a real open book which gives the same genus as that of the Heegaard surface (Theorem 3). This is not true in the setting without a real structure; in general the Heegaard genus is less than the minimum genus among Heegaard surfaces given by open books (see e.g. [24]). We also observe that there are (infinitely many) closed oriented real 3-manifolds which cannot have a *maximal* real Heegaard decomposition, i.e the real part of the real structure has less than genus+1 connected components (Proposition 6).

A *positive real stabilization* of a real open book is performed by attaching a 2-dimensional 1-handle to the page of an abstract open book and modifying the real structure and monodromy appropriately (see Section 1.2). This boils down to taking connected sum with a real tight contact S^3 , which is unique up to equivariant contact isotopy [22]. With this definition, we prove a Giroux correspondence in the existence of a real structure. We show that there is a one to one correspondence between the c_M -real contact structures on a real 3-manifold (M, c_M) up to equivariant contact isotopy and the c_M -real open books on M up to positive real stabilization (Theorem 7). As a corollary we deduce that every real 3-manifold admits a real contact structure (Corollary 9).

There are various questions that pop up after proving the basic theorems above in the real setting. To give a glimpse of that, we state some observations related to real Heegaard decompositions and real open books on lens spaces in Section 2.3. We describe some real lens spaces which cannot have a real Heegaard splitting with genus 1 or 2. Furthermore in Section 4 we present a Lefschetz fibration that does not admit a real structure, while the canonical open book of its boundary is real. We also show that this real open book cannot be filled by any real Lefschetz fibration with the same fiber topology and with arbitrary number of singular fibers (Theorem 20).

In the sequel, instead of using the term *c-real*, we usually drop the reference to c whenever the real structure is understood.

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1.1. Real open books.

Definition. Let (B, φ) be an open book decomposition of a real 3-manifold (M, c_M) . We say that (B, φ) is a *real open book decomposition*, (or shortly, a *real open book*) if $\rho \circ \varphi = \varphi \circ c_M$ where $\rho : S^1 \rightarrow S^1$ is a reflection. In particular, we have $c_M(B) = B$.

An *isomorphism* between two real open book decompositions of (M, c_M) is a pair of orientation preserving equivariant diffeomorphisms of (M, c_M) and (S^1, ρ) commuting with the projections.

As orientation reversing reflections on S^1 form a single class, we fix, once for all, $\rho(1, \theta) = (1, -\theta)$ for $(1, \theta) \in S^1 \subset \mathbb{R}^2, \theta \in [-\pi, \pi]$. By definition, any real open book decomposition of (M, c_M) has two pages, $S_0 = \varphi^{-1}(0)$, $S_\pi = \varphi^{-1}(\pi)$, which are invariant under the action of the real structure c_M . The restrictions $c_0 = c_M|_{S_0}$, $c_\pi = c_M|_{S_\pi}$ yield real structures on S_0 and, respectively, S_π . These pages together with the inherited real structures are called *real pages*. One of the fundamental property of the monodromy f of a real open book is that $f = c_0 \circ c_\pi$. Or equivalently, $f^{-1} = c_0 \circ f \circ c_0$ as well as $f^{-1} = c_\pi \circ f \circ c_\pi$. (Related to the definition above and the argumentation in the sequel, see [26].) Therefore, a real open book decomposition of (M, c_M) gives rise to a triple (S, f, c) where S is one of the real pages, c is the inherited real structure on S and f is the monodromy satisfying $f|_{\partial S} = \text{id}$ and $f^{-1} = c \circ f \circ c$. By definition, c acts on ∂S , which is the disjoint union of finitely many circles. Hence either c acts on a connected component of ∂S as reflection or c swaps a pair of connected components, reversing the induced orientations as boundary.

Lemma 1. *Let S, f, c be as above. Then the real structure $c : S \rightarrow S$ extends to a real structure on M .*

Proof: Here we explain how a real structure on a page extends to a real structure on M_f , a 3-manifold described by the abstract open book (S, f) . (As M_f is diffeomorphic to M , the real structure obtained on M_f can be pulled to M by means of a chosen diffeomorphism between M_f and M .)

First we describe the real structure on the mapping torus. Instead of the usual description, consider the mapping torus as

$$S_f = ((S \times I_+) \cup (S \times I_-)) / (x, 0) \sim (c(x), -1) \text{ and } (x, 1) \sim (f \circ c(x), 0)$$

for all x in S (see Figure 1). Here $I_+ = [0, 1]$, $I_- = [-1, 0]$.

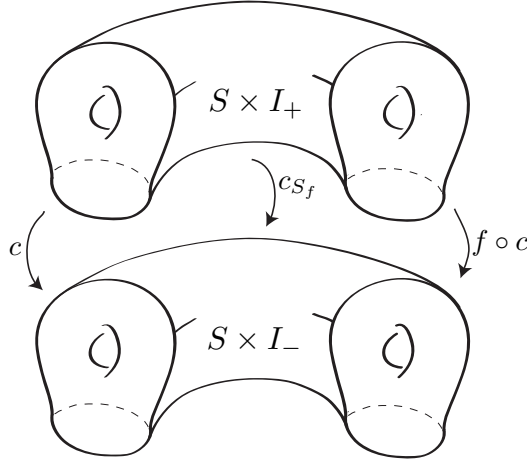


FIGURE 1. A model for a real open book.

Note that the manifold constructed has monodromy $c^{-1} \circ f^{-1} \circ c = f$.

Now consider the map $c_{S_f} : S_f \rightarrow S_f$ which acts as identity between the cylinders, i.e. (x, t) in $S \times I_+$ (resp. $S \times I_-$) is sent to $(x, t - 1)$ (resp. $(x, t + 1)$). It is an orientation preserving involution, sending the page $S_\theta = S \times \{\theta t\}$ to $S_{-\theta}$.

Restricted to each torus component of ∂S_f , the map c_{S_f} is either a rotation by π fixing exactly 4 points or it interchanges two tori components. The various ways to extend c_{S_f} as a real structure over the solid torus neighborhoods of binding components are dictated by the behavior of c on ∂S . Either a pair of solid tori are mapped to each other by c_{S_f} or c_{S_f} extends over a solid torus as rotation by π . In each case the extension is unique up to isotopy (see e.g. [16, Lemma 4.4]) and the extended map preserves the page structure. Therefore, we obtain a real structure on M_f , described by (S, f, c) , preserving the page structure. \square

Thus, we have the following definition.

Definition. Let (S, f) be an abstract open book, where S is a compact surface with boundary and $f : S \rightarrow S$ is the monodromy so that f is the identity on ∂S . Suppose that c is a real structure on S , i.e. an orientation reversing involution. An *abstract real open book* is a triple (S, f, c) with f satisfying $f \circ c = c \circ f^{-1}$.

An *isomorphism* of abstract open books is an orientation preserving diffeomorphism of the surface commuting both with the monodromy and with the real structure.

Note that once we adopt the model in Figure 1 so that c is placed on the *left*, an (embedded) real open book determines an abstract real open book uniquely up to the equivalence above.

1.2. Positive real stabilization. Recall that a positive stabilization of an abstract open book (S, f) is the abstract open book with page $S' = S \cup$ a 1-handle and with monodromy $f' \circ \tau_a$ where f' is the extension of f over the 1-handle trivially and τ_a is a right-handed Dehn twist along a curve a in S' that intersects the co-core of the 1-handle exactly once. The type of stabilizations that we need should lead a real structure on the new abstract open book. Here is the definition that will provide.

Definition. Let (S, f, c) be an abstract real open book. Let $S' = S \cup H$ where

- either H is a 1-handle with its attaching region c -invariant, (in particular if the attaching region is neighborhoods of a pair of real points, then we impose the condition that the real points belong to the same real component);
- or $H = H_1 \cup H_2$ where H_1 and H_2 are 1-handles with their attaching regions interchanged by c .

In such cases, c extends uniquely over H to a real structure, say, c' on S' . Let f' denote the extension of f over S' with $f'|_H = \text{id}$. We consider, in the former case, a simple closed curve a such that $c'(a) = a$ and that a intersects the co-core of H once (existence of such an invariant curve a is guaranteed by the imposed condition), while in the latter case, a pair of simple closed curves $a, c'(a)$ such that a and hence $c'(a)$ intersects the co-core of H_1 and, respectively, of H_2 once. Depending on the succeeding cases, let σ denote either the Dehn twist τ_a along a or the product $\tau_a \circ \tau_{c'(a)}$. Then a *positive real stabilization* of the abstract real open book (S, f, c) is defined as the open book $(S', f' \circ \sigma, c' \circ \sigma)$.

First note that $f' \circ c' = c' \circ f'^{-1}$ and $c' \circ \sigma \circ c' = \sigma^{-1}$ since c' is orientation reversing. Then it is easy to see that $(S', f' \circ \sigma \circ c' \circ \sigma)$ is really an abstract real

open book. In fact,

$$(f' \circ \sigma)(c' \circ \sigma) = f' \circ c' \circ (c' \circ \sigma \circ c') \circ \sigma = c' \circ f'^{-1} \circ \sigma^{-1} \circ \sigma = (c' \circ \sigma) \circ (\sigma^{-1} \circ f'^{-1}).$$

Furthermore the real manifold $M' = M_{f' \circ \sigma}$ is equivariantly diffeomorphic to M_f . Let us see this in the case where H is a single 1-handle (the other case is similarly treated). Observe that a sufficiently small closed neighborhood D' of the core α of H in M' is a real 3-ball, which is known to be unique up to equivariant isotopy. The local model for D can be taken as depicted in Figure 2 (see e.g [13], Section 1). If c' fixes α pointwise then the model is as in Figure 2(a) or else c' acts on α as reflection and we have the model as in Figure 2(b). A positive stabilization in the

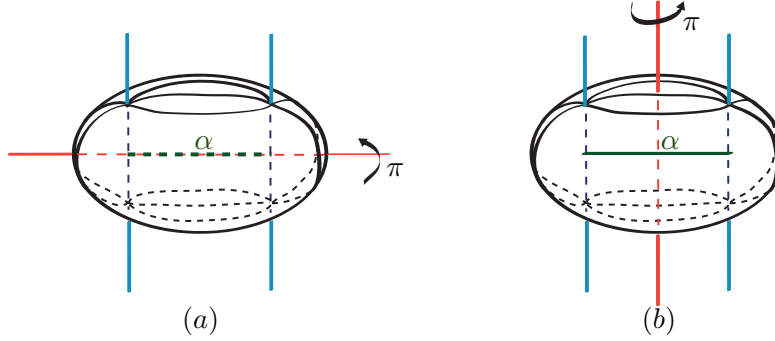


FIGURE 2. Neighborhoods of a proper embedded invariant arc on (S, c) .

non-abstract sense is nothing but excising a sufficiently small closed neighborhood D of α and gluing back a model for D' depending on the behavior of the real structure on α . Thus the uniqueness of the model in each case guarantees that the (equivariant) identity diffeomorphism from $M_f \setminus D$ to $M' \setminus D'$ uniquely extends (equivariantly) to a diffeomorphism from D to D' .

In Figure 3, we depict all possible choices for the attachment(s) of the 1-handle(s) and the extension the real structure through the handle(s). In the first six cases, the real structure acts as a reflection on the boundary component(s) of S , while in the last three cases, the real structure swaps two boundary components.

Remark 1. A positive real stabilization of type I, II, V, VII is a connected sum with real S^3 equipped with the real open book (defined by $\varphi : S^3 \setminus H \rightarrow S^1$, $\varphi(z_1, z_2) = \frac{z_1 z_2}{|z_1 z_2|}$) whose binding is the positive Hopf link $H = \{(z_1, z_2) \in S^3 : z_1 z_2 = 0\}$, where $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. This open book admits two real structures: $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$ and $(z_1, z_2) \mapsto (\bar{z}_2, \bar{z}_1)$. In the case of $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$, the real structure acts on each binding component and the real pages are as depicted at the top of Figure 4. In the other case, however, the real structure takes one binding circle to the other and acts on the real pages as shown at the bottom of Figure 4.

In [21], we showed that there is a unique tight contact 3-ball; therefore, as in the non-real case, if there exists a real contact structure *supported* (in the sense of Definition 3) by a real open book, a positive real stabilization of the open book coincides with the real contact connected sum of the manifold with real S^3 .

Other types (III, IV, VI, VIII, IX) of positive real stabilizations can be seen as connected sums with a pair of S^3 exchanged by the real structure.

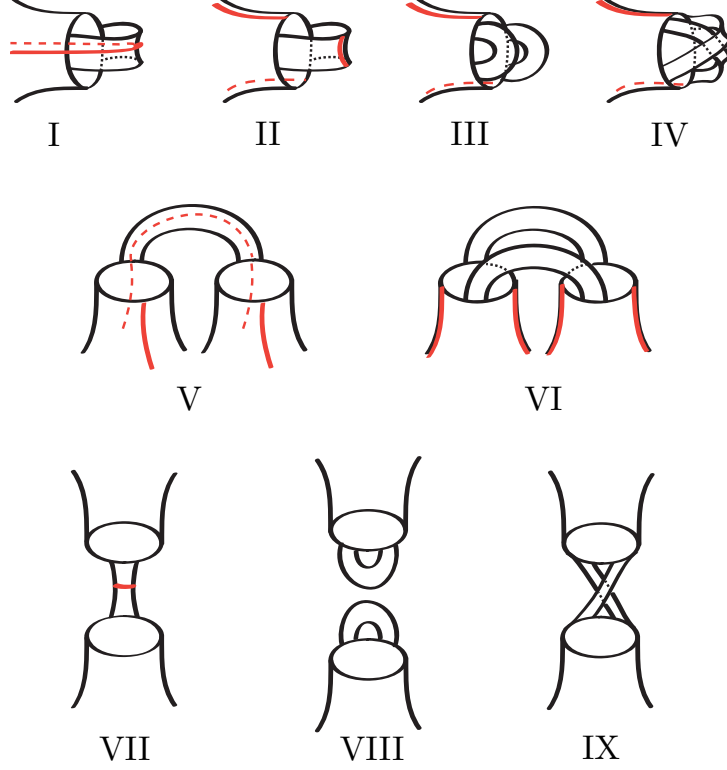


FIGURE 3. Possible handle attachments and extension of the real structure through the handle(s). The real parts are depicted in red.

1.3. Real Heegaard decompositions.

Definition. Let S be an oriented hypersurface in a closed oriented manifold M . If the topological closure of $M \setminus S$ is a disjoint union of two handlebodies diffeomorphic to each other, then the pair (M, S) is called an *embedded Heegaard decomposition*.

Let N be an oriented handlebody with nonempty boundary H and $h : H \rightarrow H$ be an orientation reversing diffeomorphism. The pair (H, h) is called an *abstract Heegaard decomposition* of the manifold $(N \cup N') / (x \sim (h(x)))$ where N' is a copy of N .

Let (M, c_M) be a real manifold and (M, S) be an embedded Heegaard decomposition. If $c_M(S) = S$ and $c_M|_S$ is orientation reversing, then we call the triple (M, c_M, S) an *embedded real Heegaard decomposition*.

It is obvious that an abstract Heegaard decomposition determines an embedded one immediately. To see the converse, let us make the following

Definition. Let (M, S) be an embedded Heegaard decomposition and $g : M \rightarrow M$ be an orientation preserving involution such that $g(S) = S$ and g is orientation reversing on S . Then we call g an *involution of the embedded Heegaard decomposition*.

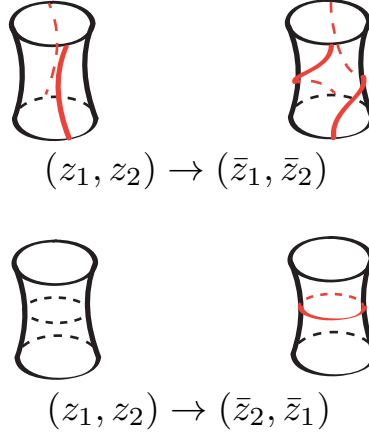


FIGURE 4. Real pages of possible real open books of S^3 whose binding is the positive Hopf link.

Provided that there is such an involution g , of course (S, g) is an abstract Heegaard decomposition for M . Existence of such an involution implies the existence of an index 2 subgroup of the mapping class group of M . Observe that for an embedded real Heegaard decomposition (M, c_M, S) with handlebodies M_j ($j = 1, 2$), if $c = c_M|_S$ is orientation reversing then (S, c) is an abstract Heegaard decomposition for M . It was proven by T. A. Nagase that every real manifold admits an embedded real Heegaard decomposition [20].

An embedded (respectively abstract) open book decomposition describes an embedded (respectively abstract) Heegaard decomposition. Let (B, φ) be an embedded real open book for the real manifold (M, c_M) . Denote by c_0 and c_π the diffeomorphisms obtained by restricting c_M to the two invariant pages F_0 and F_π . Note that $f = c_\pi \circ c_0$. Then $(M, F_0 \cup F_\pi)$ is an embedded real Heegaard decomposition and $(F_0 \cup F_\pi, c_0 \cup f \circ c_0)$, is an abstract real Heegaard decomposition for (M, c_M) .

2. REAL OPEN BOOKS IN THREE DIMENSIONS

2.1. Real open books on S^3 . Here we give a series of examples of real open book decompositions on S^3 .

It is well-known that on S^3 , up to isotopy, there is a unique real structure with nonempty real part; its fixed point set is an unknot [30] (cf. [2]). Let us fix the real structure c_0 on S^3 , considered as one point compactification of \mathbb{R}^3 , as the one induced from the rotation on \mathbb{R}^3 by π along the y -axis. We consider the open book on S^3 with binding z -axis $\cup \infty$ and with pages topologically disks. Abstractly, this is the open book of S^3 with page P_0 a disk and monodromy f_0 the identity map. There are two invariant pages on which the real structure acts as the reflection with respect to the y -axis. The corresponding Heegaard decomposition has the splitting surface S_0 a sphere and the gluing map φ_0 between the two handlebodies is a reflection on S_0 fixing an *equator*. Now let us perform a positive real stabilization of type I (see Figure 3) on the open book (P_0, f_0) to obtain the page P_1 an annulus. The monodromy f_1 is the Dehn twist τ_1 along the core a_1 of P_1 . The real structure c_1 acts as reflection with respect to the core a_1 on one invariant page and as this

reflection composed with τ_1 on the other (with respect to our convention, this page will be on the *left*); the corresponding Heegaard surface S_1 is a torus; $c_1 = \rho \circ \tau_1$ on S_1 where ρ is the reflection on $S_1 \cong T^2$ having two disjoint circles as fixed point set. The map c_1 is exactly the gluing map φ_1 between the two handlebodies. Note that c_1 is conjugate to the involution which interchanges the meridian and the longitude. In the next steps we perform positive real stabilizations of type VII and I alternatively so that we obtain a sequence of real open books and real Heegaard decompositions for S^3 : given g , there is a real Heegaard splitting with surface S_g of genus g and with the real structure and the gluing map $\varphi_g = \rho_g \circ \tau_1 \circ \tau_2 \circ \dots \circ \tau_g$ on S_g where τ_i is the Dehn twist along the curve a_i , $i = 1, \dots, g$ and ρ_g is the reflection fixing a_i , $i = 0, \dots, g$, as shown in Figure 5.

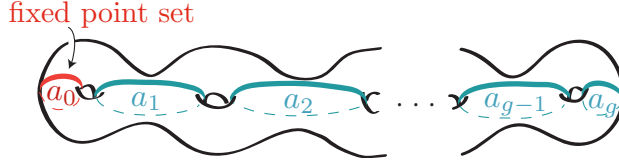


FIGURE 5. Nonseparating real structure with one real component on a surface of genus- g .

2.2. Existence of real open book decompositions. In this section we will show that every real 3-manifold admits a real open book. The proof is based on the construction of open books in three dimensions proposed by E. Giroux [11]. The construction uses the essential surfaces introduced in [9].

Suppose μ is a Morse function on M . An μ -essential surface E in M is by definition a surface on which $\mu|_E$ is still a Morse function such that its critical points on E coincide with the critical points of μ in M and the indices of the critical points on E corresponding to those in M with index 1 or 2 are 1. Not every Morse function admits an essential surface; nevertheless any Morse function can be modified to have a transversally orientable essential surface by introducing a finite number of canceling 1-, 2-handle pairs ([9], Theorem IV.2.7). Provided that the initial Morse function was self-indexing, the modification can be done in such a way that the modified Morse function is self-indexing with a single critical point of index 0, a single critical point of index 3 and g critical points of index 1 and 2 each. Here g is the genus of the new Heegaard surface, which is greater than the initial one.

Below we state the real counterpart of Theorem IV.2.7 in [9]. We prove that a real Morse function necessarily has an essential surface. Here we call a Morse function $\mu : M \rightarrow \mathbb{R}$ on (M, c_M) *real* if $\mu \circ c_M = -\mu$. For any real manifold (M, c_M) , there is a Heegaard splitting (H, φ) , where the Heegaard surface H is c_M -invariant [20]. Therefore, there exists a self-indexing real Morse function with a single minimum and maximum at levels -1 and 1 , respectively. The unique critical point with index 0 is sent to the unique critical point of index 3. Each index-1 critical point a_i is sent to an index-2 critical point b_i . Let α_i denote the intersection of $H = \mu^{-1}(0)$ with the stable surface of a_i and β_i the intersection of H with the unstable surface of b_i of the corresponding 2-handle. Then by construction, $c(\alpha_i) = \beta_i$ with $c = c_M|_H$.

Proposition 2. *Every self-indexing real Morse function on a real oriented closed 3-manifold (M, c_M) admits a transversally orientable c_M -invariant essential surface.*

Proof: The construction of a c_M -invariant essential surface around the unique critical point of index 0 and the critical points of index 1 can be done exactly as in the non-real case proposed in [9]. Namely, around the unique critical point of index 0, M is a topological 3-ball M_0 and a properly embedded 2-disk E_0 in M_0 containing the critical point is an essential surface. Let D_1 and D_2 be two disks on ∂M_0 along which the first 1-handle h_1 will be attached. We choose $\partial E_0 - (D_1 \cup D_2)$ c_M -invariant. This makes sense in the following way. The gradient-like flow of the real Morse function can be taken transversal to $M'_0 = \partial M_0 - (D_1 \cup D_2)$. Via this flow we may consider M'_0 as a subset of the real Heegaard surface H . Thus we see that M'_0 must be c_M -invariant.

Up to isotopy, any 1-handle attachment can be done in such a way that the essential surface extends through the handle. Let k be the number of 1-handles and M_i ($i = 1, \dots, k$) denote the 3-manifold with boundary obtained by attaching the i^{th} 1-handle to M_{i-1} . Then M_k is the real Heegaard surface H . Let S_i be the boundary of M_i and E_i the properly embedded essential surface in M_i ; let Γ_i denote $\partial E_i \subset S_i$. Each 1-handle h_{i+1} is attached to M_i along part of S_i . The disconnected attaching region U_{i+1} of h_{i+1} is chosen to contain two arcs lying in Γ_i . Those arcs are part of the boundary of a band B_{i+1} lying in h_{i+1} . The surface E_{i+1} is constructed naturally by attaching B_{i+1} to E_i . As above, via the gradient flow of the real Morse function, the surface $\partial M_{i+1} - U_{i+2}$ can be considered as a subset of H . In that way, it makes sense that we require ∂B_{i+1} c_M -invariant. Depending on the locus of U_{i+1} on ∂M_i and the real points on ∂h_{i+1} , B_{i+1} will sometimes have twists (as many as the real part twists along ∂h_{i+1} , see Figure 6). Note that each connected component of ∂B_{i+1} on h_{i+1} intersects cocore of h_{i+1} once (see Figure 6).

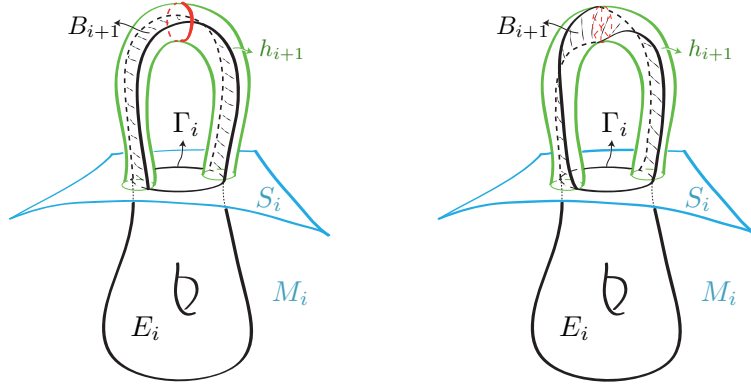


FIGURE 6. Examples of 1-handle attachments. On the left, the belt circle is real, while on the right the real structure has no real part on the 1-handle, it acts on the belt circle as the antipodal map.

By symmetry the essential surface E_k will extend to a closed essential surface in M . Furthermore this extension can be performed recursively over each 2-handle

and to get the essential surface transversally orientable. In fact, by construction Γ_k intersects each α_i ($i = 1, \dots, k$) at exactly two points. Since $c_M(\alpha_i) = \beta_i$ and Γ_k is c_M -invariant, Γ_k intersects each β_i ($i = 1, \dots, k$) at exactly two points too. Then we can extend the essential surface through a 2-handle (by attaching a band, i.e. a 2 dimensional 1-handle), see Figure 7. Note that the fact that $\#(\Gamma_k \cap \beta_i) = 2$ is essential to ease the proof; otherwise in the original proof in [9], it was required to introduce a finite number of canceling 1-, 2-handle pairs ([9], Lemma IV.2.3).

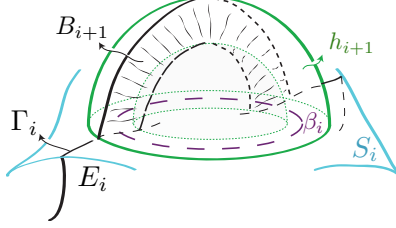


FIGURE 7. Attaching 2-handle.

After attaching all 2-handles, we obtain a manifold whose boundary is a 2-sphere on which, by the equivariant construction, the essential surface traces exactly one circle. Thus, the essential surface readily extends over the ball.

Note that by construction, the genera of H and E are equal. \square

Now we can state the main result of this section.

Theorem 3. *Every closed oriented real 3-manifold admits a real open book decomposition.*

Proof: It follows from [20, Proposition 2.4] that every real manifold (M, c_M) admits a self-indexing real Morse function μ . Moreover, by Proposition 2, μ admits an invariant essential surface E . The Heegaard splitting defined by μ is encoded by the Heegaard surface $H = \mu^{-1}(0)$ and the gluing diffeomorphism $c := c_M|_H$. Suppose $M = M_+ \cup_c M_-$ where M_{\pm} are handlebodies such that $\partial M_{\pm} = H$ and $c_M(M_+) = M_-$. On each M_{\pm} , we define a *half open book* with page $E_{\pm} = E \cap M_{\pm}$ and with binding $\Gamma = E \cap H$ by identifying M_{\pm} with the quotient space $E_{\pm} \times I_{\pm} / (x, t) \sim (x, t')$ for all $t, t' \in I_{\pm} = [0, 1]_{\pm}$ and $x \in \partial E_{\pm}$. While two handlebodies are glued along their boundaries to get M , these half open books paste together along two pages $H \setminus \Gamma$. By its construction E satisfies $c_M(E_+) = E_-$ and $c_M(\Gamma) = \Gamma$; thus, we get a real open book on (M, c) whose real pages are the halves of $H \setminus \Gamma$. \square

Let us note also the following

Corollary 4. *A closed oriented 3-manifold admits a real Heegaard decomposition if and only if it admits a real open book decomposition.*

Proof: The only if part is essentially what has been proven in Theorem 3. As for the converse, note that any real open book has two real pages which are opposite to each other. Thus, union of these pages gives the required real Heegaard surface. \square

By construction, the action of the real structure on the essential surface is orientation preserving, so we have:

Corollary 5. *Every real manifold (M, c_M) admits an embedded Heegaard splitting $M = M_1 \cup M_2$ such that $c_M(M_i) = M_i$. \square*

2.3. Examples of real Heegaard decompositions. We want to make some observations on real Heegaard decompositions over lens spaces. The classification of real structures on lens spaces up to isotopy is known [16]. In op.cit. the real structures of type C and C' on a lens space are already in the form of real Heegaard decompositions of genus 1. However on most of the lens spaces the real structures of type A , B and B' cannot be presented as a real Heegaard decomposition of genus 1 while we know that there *is* a real Heegaard decomposition [20]. Of course, in that case the higher genera decompositions will have the real structure act on the Heegaard surface not maximally. We will call a real Heegaard decomposition *maximal* if the real part of the real structure has genus+1 connected components. This leads to the following observation.

Proposition 6. *(Through an observation in [16]) There are infinitely many real closed oriented 3-manifolds which cannot have a maximal real Heegaard decomposition.*

Several more observations are in order:

- All open book decompositions with annuli pages can be made real (since the monodromy can only be a power of the Dehn twist along the belt circle and any power of the Dehn twist can be written as a product of two real structures). Thus, all lens spaces admitting an open book with annuli pages can be made real and hence admit a real Heegaard splitting of genus 1. For example, in the link of A_n singularities (the lens spaces $L(p, p-1)$ so that the coefficients of the continued fractions of $-p/q$ are only -2 's) the *horizontal* open book obtained from the plumbing (see e.g. [23] for the construction) has annuli pages.
- The lens spaces $L(p, q)$ where the coefficients of the continued fraction expansion of $-p/q$ are all -2 except one which is -3 have an open book decomposition with page thrice punctured sphere and with monodromy a product of certain powers of the Dehn twists around the boundary components. Since with respect to any real structure on the page the boundary components are invariant, these open books are always real for any choice of real structure. Hence, this type of lens spaces have a real genus-2 Heegaard splitting.
- For an example to above consider $L(5, 3)$ on which there are 3 real structures; they are of types A , B and B' . Hence there is no genus-1 real Heegaard decomposition of $L(5, 3)$. Meanwhile, on a sphere with three punctures there are only two real structures. Thus, with respect to at least one real structure the corresponding Heegaard decomposition must have genus greater than 2.

Following this discussion, we ask a natural question:

Question. Given a real closed oriented 3-manifold (M, c_M) , what is the c_M -real Heegaard genus, i.e. what is the minimum genus among all real Heegaard decompositions representing c_M ?

3. REAL GIROUX CORRESPONDENCE

In this section we prove

Theorem 7. (*Real Giroux Correspondence*) *Let (M, c_M) be a real 3-manifold. Then there is a one to one correspondence between the real contact structures on M up to equivariant contact isotopy and the real open books on M up to positive real stabilization.*

The proof of the theorem follows from Propositions 8, 10, 11 and 14.

First we explain how to construct a real contact form on the real manifold (M_f, c_{M_f}) built-up as before from the abstract real open book decomposition (S, f, c) . The construction is the real version of the one of W. P. Thurston and H. Winkelnkemper [29] and it motivates the following definition.

Definition. We say that a real contact structure $\xi = \ker \alpha$ on a real 3-manifold (M, c_M) is *supported* by a real open book decomposition (B, p) of (M, c_M) if

- (1) the 2-form $d\alpha$ induces a symplectic form on each page, defining its positive orientation,
- (2) the 1-form α induces a contact form on the binding B .

Proposition 8. *Every real open book decomposition (S, f, c) supports a real contact structure $\xi = \ker \alpha$ on $M_f = S_f \cup (\bigsqcup_{|\partial S|} S^1 \times D^2)$.*

Proof: Let β be a 1-form on S such that $\beta = e^*d\theta$ near ∂S and $d\beta$ is a positive volume form on S . It is elementary to show that such a form exists. An argument can be found in [7]. We now consider the 1-forms $\beta_K^- = \beta - Kdt$ on $S \times I_-$ and $\beta_K^+ = (1-t)c^*\beta + t(f \circ c)^*\beta + Kdt$ on $S \times I_+$ so that we define

$$\hat{\beta}_K = \begin{cases} \beta_K^+ & \text{if } t \in I_+, \\ \beta_K^- & \text{if } t \in I_-, \end{cases}$$

which, by definition, induces a 1-form on the mapping torus

$$S_f = ((S \times I_+) \cup (S \times I_-)) / \sim_{(x, 0_+) \sim (c(x), 1_-)} \text{ and } \sim_{(x, 1_+) \sim (f \circ c(x), 0_-)}$$

However, this form is in general not real. Thus let us set $\alpha_K = \hat{\beta}_K - c_{S_f}^*(\hat{\beta}_K)$. By definition α_K is a real form. As the action of c_{S_f} is the identity on $S \times I_{\pm}$, we get

$$\alpha_K = \begin{cases} \beta_K^+ - \beta_K^- & \text{if } t \in I_+, \\ \beta_K^- - \beta_K^+ & \text{if } t \in I_-. \end{cases}$$

We claim that for large values of K , α_K is a real contact form on S_f . To prove this claim we only need to show that $\beta_K^+ - \beta_K^- \wedge d(\beta_K^+ - \beta_K^-)$ is positive on S_f for sufficiently large K where positivity is determined with respect to the orientation defined by the form $d\beta \wedge dt$. We have

$$\begin{aligned} \beta_K^+ - \beta_K^- \wedge d(\beta_K^+ - \beta_K^-) &= 2Kdt \wedge [(1-t)dc^*\beta + td(f \circ c)^*\beta - d\beta] \\ &\quad + dt \wedge [(f \circ c)^*\beta \wedge c^*\beta + \beta \wedge ((f \circ c)^*\beta - c^*\beta)] \\ &\quad + [(1-t)c^*\beta + t(f \circ c)^*\beta + \beta] \\ &\quad \wedge [(1-t)dc^*\beta + td(f \circ c)^*\beta - d\beta]. \end{aligned}$$

The last term evaluates zero on S_f . This is because both β and $d\beta$ are forms on the surface S , so are their pull-backs by a diffeomorphism of S . Thus any wedge product of the form $x \wedge dx, x \in \{\beta, c^*\beta, (f \circ c)^*\beta\}$ evaluated on the 3-manifold S_f is zero. In addition, as c is orientation reversing, both $-d(c^*\beta) = -c^*d\beta$ and

$-d(f \circ c)^*\beta = -(f \circ c)^*d\beta$ are volume forms on S defining the same orientation as $d\beta$. Thus, the first term $2Kdt \wedge [(1-t)c^*d\beta + td(f \circ c)^*\beta - d\beta]$ which is equal to $2K[-(1-t)c^*d\beta - td(f \circ c)^*\beta + d\beta] \wedge dt$ is positive on S_f and is dominant when K is large enough.

For the extension of α_K over the solid tori, we need to consider two separate cases which are distinguished by the action of c_S on ∂S . As discussed above, either c_S acts on a boundary component as reflection or it switches two boundary components.

Case 1: Let S_∂^1 denote the boundary component of ∂S on which c_S acts as reflection.

We consider $\nu(S_\partial^1) = \{(s, \theta) : s \in [-\epsilon, 0], \theta \in [-\pi, \pi]\}$ where $c_S|_{\nu(S_\partial^1)}(s, \theta) = (s, -\theta)$. On the other hand, we take $S^1 \times D^2 = \{(\vartheta, r, \varphi) : \vartheta, \varphi \in [-\pi, \pi], r \in [0, 1]\}$ together with the real structure $c_{S^1 \times D^2}(\vartheta, r, \varphi) = (-\vartheta, r, -\varphi)$ and the 1-form $\alpha'_r = h_1(r)d\vartheta + h_2(r)d\varphi$. Note that, α'_r is a real form for all r and is a contact form for those r satisfying $h_1h'_2 - h'_1h_2 > 0$.

We identify $S^1 \times [1 - \epsilon, 1] \times S^1 \subset S^1 \times D^2$ with $\nu(S_\partial^1) \times S^1$ by an equivariant orientation preserving diffeomorphism Υ defined as

$$\begin{aligned} \Upsilon : S^1 \times [1 - \epsilon, 1] \times S^1 &\rightarrow \nu(S_\partial^1) \times S^1 \\ (\vartheta, r, \varphi) &\mapsto (1 - r - \epsilon, \vartheta, \varphi). \end{aligned}$$

Thus, we get

$$\Upsilon^*(\alpha_K|_{\nu(S_\partial^1) \times S^1}) = \begin{cases} -2e^{1-r-\epsilon}d\vartheta + 2Kd\varphi & \text{if } \varphi \in [0, \pi], \\ 2e^{1-r-\epsilon}d\vartheta - 2Kd\varphi & \text{if } \varphi \in [-\pi, 0]. \end{cases}$$

Since we require the extended 1-form to be positive on the binding, and to match with α_K on $S^1 \times [1 - \epsilon, 1] \times S^1$, it is enough to find smooth h_1 and h_2 such that

$$\begin{aligned} (1) \quad & h_1(r) = 1 \text{ and } h_2(r) = r^2 && \text{near } r = 0 \\ (2) \quad & h_1(r) = -2e^{1-r-\epsilon} \text{ and } h_2(r) = 2K && \text{near } r = 1 \\ (3) \quad & h_1(r)h'_2(r) - h'_1(r)h_2(r) > 0 && \forall r \in [0, 1], \end{aligned}$$

and it is easy to see that there exist h_1, h_2 satisfying the above conditions.

Case 2: Let $S_{\partial_1}^1$ and $S_{\partial_2}^1$ be two boundary components of S such that $c_S(S_{\partial_1}^1) = S_{\partial_2}^1$. We consider two contact solid tori $S^1 \times D^2$ with contact structures $\alpha'(r) = h_1d\vartheta + h_2(r)d\varphi$ and $c_{S^1 \times D^2}^*(\alpha'(r)) = -h_1(r)d\vartheta - h_2(r)d\varphi$.

As before, we identify $S^1 \times [1 - \epsilon, 1] \times S^1 \subset S^1 \times D^2$ with $\nu(S_{\partial_1}^1) \times S^1$ by Υ then the identification Υ' of $\nu(S_{\partial_2}^1) \times S^1 = c_{S_f}(\nu(S_{\partial_1}^1)) \times S^1$ is determined by the symmetry

$$\begin{array}{ccc} S^1 \times [1 - \epsilon, 1] \times S^1 & \xrightarrow{\Upsilon} & \nu(S_{\partial_1}^1) \times S^1 \\ \downarrow c_{S^1 \times D^2} & & \downarrow c_{S_f} \\ S^1 \times [1 - \epsilon, 1] \times S^1 & \xrightarrow{\Upsilon'} & \nu(S_{\partial_2}^1) \times S^1. \end{array}$$

Near $S_{\partial_1}^1$ we extend α_K as in the previous case and the extension of α_K near $S_{\partial_1}^1$ is obtained symmetrically, using $-h_1$ and $-h_2$. \square

From Theorem 3 and Proposition 8 we get

Corollary 9. *Every closed oriented real 3-manifold admits a real contact structure.*

\square

Proposition 10. *Two real contact structures supported by the same real open book decomposition are equivariantly isotopic.*

Proof: The existence proof of the proposition with no equivariance requirement applies with slight modifications to equivariant case.

Let $\xi_0 = \ker \alpha_0$ and $\xi_1 = \ker \alpha_1$ be two real contact structures supported by the same real open book decomposition. We first isotope $\alpha_i, i \in \{0, 1\}$ near binding.

Let $S^1 \times D_\epsilon^2 = \{(\vartheta, r, \varphi) : r \in [0, \epsilon], \varphi, \vartheta \in [-\pi, \pi]\}$ be a neighborhood of binding, and $h(r)$ a function such that $h(r) = r^2$ near 0, and $h = 1$ near ϵ and $h'(r) > 0, \forall r$. Define $\alpha_{i,R} = \alpha_i + Rh(r)d\varphi, R \geq 0$. It can be easily checked that all $\alpha_{i,R}$ for $i \in \{0, 1\}$ and $R \geq 0$ are contact and real, thus they define (equivariantly) isotopic real contact forms.

Finally, we consider $\alpha_{t,R} = (1-t)\alpha_{0,R} + t\alpha_{1,R}$ which is a real form for any $t \in [0, 1]$ and $R \geq 0$ and a contact form for any t and for sufficiently large R . \square

Proposition 11. *Every real contact structure on a closed 3-dimensional real manifold is supported by a real open book.*

Proof follows from Lemma 12 and Lemma 13. First we construct a *real contact cell decomposition* on M (Lemma 12) and then using that we build the real open book (Lemma 13).

Let (M, ξ, c_M) be the real contact manifold. A *real contact cell decomposition* over M is a cell decomposition of M with each k -skeleton c_M -invariant ($k = 0, 1, 2, 3$); each 2-cell is convex with $\text{tw}(\partial D, D) = -1$; and ξ is tight when restricted to each 3-cell.

Lemma 12. *Every closed real contact 3-manifold (M, ξ, c_M) has a real contact cell decomposition.*

Proof: We cover M with a c_M -invariant, finite set of Darboux balls. We then choose a real cell decomposition such that each 3-cell lies in the interior of a Darboux ball. Note that every real manifold has an equivariant cell decomposition [17].

We now turn this equivariant cell decomposition into a contact one. First, we make the 1-skeleton Legendrian. Note that the real structure acts on a 1-cell as the identity or a reflection, or it swaps two 1-cells. By definition, the real part is Legendrian. If the real structure exchanges two 1-cells, those cells are disjoint. If their closure are not disjoint, by possibly subdividing the cell decomposition equivariantly, we can make sure that they have a single common end point which is real. As the condition of being Legendrian is satisfied at a real point, we can make such a 1-cell pair Legendrian symmetrically. If the real structure acts on a 1-cell e^1 as reflection then the two halves of e^1 can be perturbed Legendrian symmetrically relative their common real end point, at which the Legendrian condition is satisfied.

As each 2-cell remains inside the Darboux balls, (weak) Bennequin inequality applies, so $\text{tw}(\partial e^2, e^2) \leq -1$ for each 2-cell e^2 . Hence we can perturb each e^2 to make it convex [15, Proposition 3.1]. The real structure can act on a 2-cell as a reflection or a rotation, or it switches two 2-cells. In the latter case, e^2 and $c_M(e^2)$ are disjoint; closure of them may have a common boundary piece which is real. Therefore each such 2-cell e^2 can be perturbed relative its boundary to make it convex and $c_M(e^2)$ can be perturbed symmetrically. If the real structure acts on e^2 as reflection, then as before the two halves can be perturbed symmetrically relative their boundary to make them convex. If c_M acts on a 2-cell e^2 as rotation,

then the characteristic field on e^2 is an anti-symmetric vector field tangent to the boundary. The work in [3] shows that among all such vector fields, the structural stable ones are dense. Thus we conclude that e^2 can be perturbed convex, employing similar final steps in the proof of Proposition 3.3 in [21]. Once every 2-cell is made convex, we subdivide those 2-cells with $\text{tw}(\partial e^2, e^2) = -n < -1$ to get 2-cells with correct twisting. The subdivision can be made symmetrically. This is true even if the 2-cell is c_M -invariant since the dividing set can be chosen invariant [21, Theorem 3.1]. By the Legendrian realization principle [15] each arc added to 2-cells during the subdivision can be made Legendrian keeping the 2-cells convex. Therefore, we obtain a c_M -equivariant contact cell decomposition on M . \square

Given (M, ξ, c_M) take a real contact cell decomposition. The *real ribbon* of the 1-skeleton G is a compact surface R with boundary such that (i) c_M acts on R reversing the orientation, (ii) R retracts onto G and (iii) R is tangent to ξ_p if and only if $p \in G$. It is easy to see that any invariant Legendrian graph has a real ribbon.

Lemma 13. *The boundary B of a real ribbon R in (M, c_M) is the binding of a real open book that supports the real contact structure ξ .*

Proof: Note that the second claim is independent from the real structure and a proof has been given in [6]. Moreover, since it is known that the ribbon is a page of an open book supporting ξ (see [9] or [6]), we only need to show that the open book having real ribbon as a page can be constructed equivariantly.

To this end, let us first choose an invariant neighborhood $N(R)$ of the real ribbon R such that $\partial R \subset \partial N(R)$ and that the boundary $\partial N(R)$ is a convex surface with dividing curve ∂R . We can identify $N(R)$ with $R \times [-1, 1]$ quotient by $(r, t) \sim (r, t')$, for $r \in \partial R$ and $t, t' \in [-1, 1]$ on which real structure acts as $(r, t) \mapsto (c_M|_R(r), -t)$. Then the projection $N(R) \equiv R \times [-1, 1] / \sim \rightarrow [-1, 1]$ defines one “half” of the open book. By the construction of the contact cell decomposition each 2-cell D of the cell decomposition is convex and has boundary with twisting -1 relative D . Since R twists with the contact structure along the 1-skeleton, each 2-cell D intersects ∂R at exactly two points. Thus, we can extend the fibration in $N(R)$, in an obvious way, in a neighborhood of each 2-cell. In Figure 8 (left), we depict a 2-cell foliated by the pages of the open book. Each page traces an arc and all arcs meet at points touching ∂R .

As for the extension of the real structure, let us first assume that the 2-cell D is invariant, so the boundary of the 2-cell admits an involution which is a reflection or a rotation. In the case of reflection, the involution fixes the points touching ∂R . Since any two involutions on a circle fixing the same pair of points can be extended to an involution of a disk in such way that the extension is unique up to isotopy relative to boundary, we can extend the real structure on a neighborhood of the 2-cell in such a way that it commutes with the fibration (Figure 8, middle). In the case of rotation the points touching ∂R interchanged by the real structure, so similar idea applies (Figure 8, right). The case when the real structure swaps two 2-cells can also be similarly treated. Now, if we trace the dividing curve on the boundary of each $(3\text{-cell} - \text{int}(N(R)))$, we obtain a connected closed curve [12]. Therefore, we can extend the open book over 3-cells in such a way that the dividing curve becomes a part of the binding and the pages foliate the 3-cells as shown in Figure 9. In fact, an involution on S^2 with the prescribed image of a fixed closed

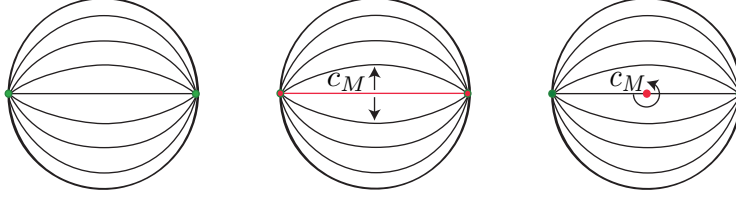


FIGURE 8. A 2-cell foliated by the pages of the open book (left), and possible actions of a real structure on a foliated 2-cell (middle, right).

curve can be extended to a unique (up to isotopy relative boundary) involution of B^3 preserving the fibration described above. \square

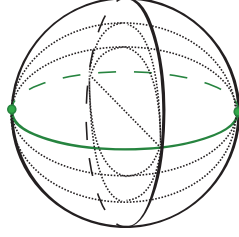


FIGURE 9. A 3-cell foliated by the pages of the open book.

To finish the proof of Real Giroux Correspondence, we prove the following

Proposition 14. *Two real open books on the real contact manifold (M, ξ, c_M) supporting the real contact structure ξ are related by positive real stabilizations.*

Proof: The proof follows the idea of the non-real case proposed by E. Giroux [10, 11]. It is enough to show that any real open book supporting ξ comes from a real contact cell decomposition after possibly a number of positive real stabilizations. Then the rest of the proof will follow similar lines of the original proof.

Let (B, φ) be a real open book of (M, c_M) supporting ξ . We consider the Heegaard decomposition $M = M_1 \cup_H M_2$ defined by φ with the Heegaard surface $H = \varphi^{-1}(0) \cup_B \varphi^{-1}(\pi)$. Let c be the pasted diffeomorphism $c_0 \cup c_\pi$ where c_θ ($\theta = 0, \pi$) is the inherited real structure on the page $S_\theta = \varphi^{-1}(\theta)$. Recall that $c_\pi = f \circ c_0$ where f is the monodromy of the open book. Note that H is convex; in fact, consider a self-indexing real Morse function $\mu : M \rightarrow [-1, 1]$ with $\mu^{-1}(0) = H$ and $M_1 = \mu^{-1}([-1, 0])$, $M_2 = \mu^{-1}([0, 1])$. Then its gradient field is by definition contact and transversal to H , rendering H convex. Moreover B is an invariant dividing curve on H .

Let Σ be an abstract surface diffeomorphic to a page. We consider the abstract handlebody $W = \Sigma \times [0, 1] / (x, t) \sim (x, t')$ for all $x \in \partial\Sigma$ and $t, t' \in [0, 1]$. Note that $\Sigma_0 = \Sigma \times \{0\}$ inherits a real structure c_0 from S_0 (keeping the same notation for the real structures on S_0 and on Σ_0). We define a cell decomposition on W as follows. Consider a c_0 -invariant set A of disjoint proper arcs on Σ_0 dividing it into disks.

Each such arc α defines a disk $D_\alpha = \alpha \times [0, 1] / \sim \subset W$. By the choice of the set A , the set of all D_α 's cuts W into 3-balls, so we have a cell decomposition on W whose 1-skeleton is two copies of A (one on Σ_0 , the other, say A' on $\Sigma_1 = \Sigma \times \{1\}$) and whose 2-skeleton consists of the disks $\{D_\alpha\}_{\alpha \in A}$ and the disks on $\partial W \setminus \{A \cup A'\}$.

Let C denote a c_0 -invariant set of closed curves on Σ_0 forming a core for Σ_0 . We may impose that C contains the closed real curves (if exist) lying on Σ_0 . Such a core can always be chosen. Now, we add C to the 1-skeleton of the cell decomposition described above to get a cell decomposition on W in the sense of L. Siebenmann [28].

The cell decomposition on W constructed above give a cell decomposition on M (by means of the identifications of W with M_i) whose 1-skeleton contains $A \cup C$ on S_0 and $A' \cup f(A')$ on S_π . Note first that, the 1-skeleton is c_M -invariant. Namely, we have $c_0(C \cup A) = C \cup A$, as C and A are chosen c_0 -invariant. Meanwhile, $c_\pi(A') = f \circ c_0(A') = f(A')$; thus, $A' \cup f(A')$ is c_π -invariant. It is easy to see that by construction, 2- and 3-skeletons are also c_M -invariant. Thus, we get a real cell decomposition on (M, c_M) .

We now perturb 1-skeleton equivariantly as before to make it Legendrian. For 1-cells on S_0 Legendrian realization principle applies, so we make them Legendrian keeping S_0 convex and the 1-cells c_0 -invariant. Then we perturb the 1-cells on S_π Legendrian keeping them equivariant. Note that we are not required to reside on S_π nor to keep S_π convex here. Again as before, each 2-cell has twisting less than -1 , so for each 2-cell e^2 we can arrange $\text{tw}(\partial e^2, e^2)$ to be -1 by subdividing e^2 . Each arc of the subdivision can be made Legendrian keeping e^2 convex. By construction each 3-cell is away from a core of a page, thus it lies in a tight manifold [11].

Now, for each arc in the 1-skeleton not lying on S_0 , we can perform a stabilization to include it to the page S_0 . Such a stabilization exists since by the construction of the cell decomposition, each such arc together with an arc on S_0 bounds a disk D with $\text{tw}(\partial D, D) = -1$. Hence, we perform a finite number of positive real stabilizations so that the stabilized S_0 becomes a real ribbon of the 1-skeleton. \square

It is worth mentioning that in the proof of Lemma 12, it is possible to start with an equivariant cell decomposition where the real part lies entirely in the 1-skeleton. Furthermore, it is easy to observe that if given an equivariant cell decomposition, one can always get the real part lie entirely in the 1-skeleton. Then by turning such a decomposition into a contact cell decomposition and by defining the real ribbon, we indeed obtain a real open book with real part lying entirely on a page. However, if we restrict ourselves to such class of equivariant cell decompositions, then the proof for Proposition 14 would not work. More precisely, it is not true that given a real open book there is a series of positive real stabilizations which produces a real open book with the real part lying entirely on a page. For example, the real Hopf open book depicted on the top of Figure 4 cannot be positively equivariantly stabilized so that the real part becomes a subset of a page. In fact, it is not hard to see that if a page contains more than one piece of the same connected component of the real part, then there is no way to get the real part lie on a page by positive real stabilizations.

4. REMARKS ABOUT FILLING REAL OPEN BOOKS BY REAL LEFSCHETZ FIBRATIONS

There is a very close relation between open book decompositions and Lefschetz fibrations over D^2 , with fibers surfaces with boundary. It is known that any open book decomposition is filled by a Lefschetz fibration if its monodromy can be factorized as a product of positive Dehn twists. Moreover, two fibrations filling the same open book are isomorphic if and only if the two such factorizations are Hurwitz equivalent. In this section, we investigate relationship between real open books and real Lefschetz fibrations. We present an example of a real open book which cannot be filled by a real Lefschetz fibration, despite the fact that it is filled by non-real Lefschetz fibrations.

Let E be an oriented smooth 4-manifold and B an oriented smooth surface. A *genus- g Lefschetz fibration* of E is a proper smooth projection $p : E \rightarrow B$ such that p has only finitely many critical points in $\text{int}(E)$ with pairwise distinct images around which one can choose complex charts such that the projection takes the form $(z_1, z_2) \rightarrow z_1^2 + z_2^2$. Moreover, the inverse image of a regular value is a closed oriented smooth surface of genus g . It follows from the definition that if $\partial B \neq \emptyset$, then $\partial E = \pi^{-1}(\partial B)$ is a fiber bundle over ∂B . The notion of Lefschetz fibration can be slightly generalized to cover the case of fibers with boundary. Then E turns into a manifold with corners and its boundary, ∂E , becomes naturally divided into two parts: $p^{-1}(\partial B)$ and $\partial F \times B$. In this case, if, in particular, $B = D^2$, then ∂E admits a canonical open book decomposition with binding $p^{-1}(0)$ and projection $p|_{\partial E}$. A *real Lefschetz fibration* is a fibration together with a pair of real structures $c_E : E \rightarrow E$ and $c_B : B \rightarrow B$ commuting with the fiber structure.

It is known that around a singular fiber, Lefschetz fibrations are determined by the monodromy that is a single positive Dehn twist around a simple closed curve, the *vanishing cycle*. Algebraically, Lefschetz fibrations over $B = D^2$ can be encoded by the factorization $(t_{a_1}, \dots, t_{a_n})$ of the monodromy $f = t_{a_n} \circ \dots \circ t_{a_1}$ (because of the composition notation the order is reversed) along ∂D^2 into a product of positive Dehn twists considered up to Hurwitz equivalence. Namely, two such factorizations are called *Hurwitz equivalent* if one can get from one factorization to the other by a finite sequence of Hurwitz moves:

$$\begin{aligned} (\dots, t_{a_i}, t_{a_{i+1}}, \dots) &\rightarrow (\dots, t_{a_i}^{-1} \circ t_{a_{i+1}} \circ t_{a_i}, t_{a_i}, \dots), \\ (\dots, t_{a_i}, t_{a_{i+1}}, \dots) &\rightarrow (\dots, t_{a_{i+1}}, t_{a_{i+1}} \circ t_{a_i} \circ t_{a_{i+1}}^{-1}, \dots); \end{aligned}$$

and possibly a global conjugation.

In what follows, we consider a genus-1 Lefschetz fibration $p : X \rightarrow D^2$ with exactly two singular fibers. Choose a base point $d \in \partial D^2$ and consider a basis (γ_1, γ_2) of $\pi_1(D^2 \setminus \{\text{critical values}\}, d)$ obtained by connecting the base point d to the positively oriented simple loops, each surrounding the corresponding critical value once. Fix an identification of $F_d = p^{-1}(d)$ with $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Denote by a , the class on T^2 of $(1, 0) \in \mathbb{R}^2$ and by b , the class of $(0, 1)$ and consider the curves u, v represented respectively by $3a + 5b$ and a on T^2 . We assume that the monodromy along γ_1 and γ_2 are given respectively by the positive Dehn twists t_u and t_v . In other words, the corresponding singular fibers are obtained from T^2 by pinching the curves u and v . The total monodromy of the fibration, thus, becomes the composition $f = t_v \circ t_u$.

Recall that $f \mapsto f_*$ defines an isomorphism from the mapping class group $\text{Map}(T^2)$ (the identity component of the space of diffeomorphisms) of the torus to the group of automorphisms of $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}a \oplus \mathbb{Z}b$. The latter is isomorphic to

$$\text{SL}(2, \mathbb{Z}) = \left\{ [t_a] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, [t_b] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} : \begin{array}{l} [t_a][t_b][t_a] = [t_b][t_a][t_b] \\ ([t_a][t_b])^6 = \text{id} \end{array} \right\}$$

where $[t_o]$ refers to the matrix representation of the automorphism t_{o*} . With respect to the above presentation, $[f] = [t_v][t_u]$ is given by the matrix $\begin{bmatrix} -39 & 25 \\ -25 & 16 \end{bmatrix}$.

Proposition 15. $p : X \rightarrow D^2$ does not admit a real structure.

Proof: Suppose that $p : X \rightarrow D^2$ admits a real structure. That is to say, there exist real structures $c_X : X \rightarrow X$ and $c_{D^2} : D^2 \rightarrow D^2$ such that $p \circ c_X = c_{D^2} \circ p$. Therefore, the critical points as well as their images are invariant under the action of the real structures c_X and c_{D^2} , respectively, so either they are both real or they are interchanged by the real structures. By definition of real Lefschetz fibrations, the decomposition (t_u, t_v) associated to (γ_1, γ_2) is Hurwitz equivalent to a decomposition $(t_{c_X(v)}, t_{c_X(u)})$ associated to $(c_{D^2}(\gamma_2), c_{D^2}(\gamma_1))$. Although, the positions of (γ_1, γ_2) can be arbitrary with respect to the real structure c_{D^2} , by conjugation by a power of f (such a conjugation preserves Hurwitz classes) and by moving, if necessary, the base point on the boundary, we can assume that we have only the two positions shown in Figure 10. In the former case (the case shown on the left in Figure 10), the fiber F_d can be endowed with a real structure c by pulling a real structure on a real fiber between the two real singular fibers. Up to isotopy, we can assume that vanishing cycles u, v are invariant under the action of the real structure c . Lemma 16 concerns the intersection number of invariant curves and prohibits this case, (since the intersection number of u and v is equal to 5).

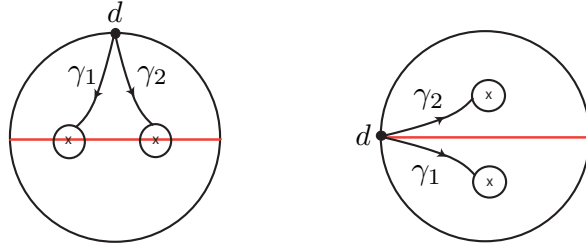


FIGURE 10. Two possible positions of (γ_1, γ_2) on base D^2 regarding the real structure reflection with respect to horizontal.

Now assume that critical points are exchanged by the real structure (this is the case depicted on the right in Figure 10). Let $c : T^2 \rightarrow T^2$ be the real structure conjugate to the inherited real structure on the base fiber F_d . Then, by the assumption we have $c(u) = v$; as a consequence, $u + v$ and $u - v$ are elements of \pm eigenspaces H_+^c and, respectively, H_-^c of $c_* : H_1(T^2, \mathbb{Z}) \rightarrow H_1(T^2, \mathbb{Z})$. Thus, the primitive forms of the classes of $u + v$ and $u - v$ form a basis, respectively of H_+^c and H_-^c . Moreover, for any real structure c , the intersection number of the bases of H_+^c and H_-^c is at most 2 which is violated by our choice of u and v . \square

Lemma 16. *The number of intersection points of two invariant curves on a real torus can only be 0, 1 or 2.*

Proof: The real structure restricted to an invariant curve defines an action on the curve. This action can be the identity, a reflection or an antipodal involution; let us call those curves a real curve, a reflection curve and an antipodal curve respectively. Recall that up to equivariant diffeomorphisms, the real structure on a torus are distinguished by the number of real components that can be 0, 1 or 2. for each real structure we have

- if c has no real components, then there exist two c -equivariant isotopy classes of antipodal curves and no classes of other types;
- if c has one real component, then T^2 contains a unique c -equivariant isotopy class of non-contractible reflection curves, a unique class of antipodal curves, and a unique real curve;
- if c has two real components, then T^2 contains no c -equivariant isotopy class of antipodal curves, a unique class of reflection curves, and two classes of real curves.

The result follows from the following observations: a reflection curve has two real points, so the intersection with a real curve must be at 0, 1 or 2 points. Moreover, a reflection curve must intersect an antipodal curve at 0 or 2 points, and real curves and antipodal curves are disjoint. \square

Proposition 17. *Let $M = \partial X$; then M as a surface bundle is real.*

Proof: By [26, Proposition 4], it is enough to check whether its monodromy is real, i.e. it admits a decomposition into a product of two real structures.

Real structures on T^2 correspond to involutive elements of determinant -1 in $\mathrm{GL}(2, \mathbb{Z})$. Here we have $[f] = \begin{bmatrix} -39 & 25 \\ -25 & 16 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 8 & -5 \end{bmatrix} \begin{bmatrix} -120 & 77 \\ -187 & 120 \end{bmatrix}$, a product of two involutive elements of determinant -1 . \square

Remark 2. In [26], an explicit classification of real elements of $\mathrm{SL}(2, \mathbb{Z})$ as well as a particular decomposition of a product of two real structures for each conjugacy class is given. According to [26, Proposition 4], a hyperbolic element, i.e. an element with absolute value of trace greater than 2, of $\mathrm{SL}(2, \mathbb{Z})$ is real if and only if its *cutting period cycle* $[a_1, \dots, a_{2k}]$ is odd-bipalindromic. For hyperbolic matrices, the cutting period cycle together with the sign of the trace is a complete invariant of conjugacy classes and it can be recovered from the trace and the periodic tail of the continued fraction expansion of the slope of the eigenvectors. In our specific example the cutting period cycle of f is found to be [1313] which is odd-bipalindromic, i.e. up to cyclic ordering, it has two palindromic pieces 1, 313 of odd length.

So far we have constructed a T^2 -fibered Lefschetz fibration cannot be real while the T^2 bundle at its boundary is real. Now we show that a similar construction can be cooked up to obtain a real open book at the boundary.

Denote by $\check{p} : \check{X} \rightarrow D^2$ the Lefschetz fibration with boundary obtained from $p : X \rightarrow D^2$ above by taking out a neighborhood of a section. Note that such a section always exists for genus 1 Lefschetz fibrations [19]. The regular fibers of \check{p} are tori with one boundary component. The fibration \check{p} has no real structure, since otherwise p would have one. However, we have:

Proposition 18. *The canonical open book of $\check{M} = \partial\check{X}$ admits a real structure.*

Proof: As a consequence of Lemma 1, it is enough to show that the monodromy of $\check{p}|_{\check{M}}$ is real. Note that the monodromy of $\check{p}|_{\check{M}}$ is an element of $\text{Map}(T^2 \setminus \text{int}D^2, \partial)$, the group of relative isotopy classes of orientation preserving diffeomorphisms which are identity on the boundary.

The crucial observation is that being real in $\text{Map}(T^2)$ is equivalent to being real in $\text{Map}(T^2 \setminus \text{int}D^2, \partial)$. Namely, there is a short exact sequence

$$0 \rightarrow \langle t_\partial \rangle \rightarrow \text{Map}(T^2 \setminus \text{int}D^2, \partial) \rightarrow \text{Map}(T^2) \rightarrow 0$$

given by the central extension of $\text{Map}(T^2)$, where t_∂ is the Dehn twist along the boundary component. Obviously, images of real elements of $\text{Map}(T^2 \setminus \text{int}D^2, \partial)$ are real. For the converse, suppose $\text{im}[f] = [c][c']$ for some $[f] \in \text{Map}(T^2 \setminus \text{int}D^2, \partial)$ and real structures $[c], [c'] \in \text{GL}(2, \mathbb{Z})$. If necessary, by replacing c by $c \circ g$ and c' by $g^{-1} \circ c'$ for some diffeomorphism g , we can assume that c and hence c' leaves ∂T^2 invariant. We have $t_\partial = c' t_\partial^{-1} c'$. Thus $t_\partial^{-1} c$ is a real structure which we denote by c'' . As $[f] = [c][c'] t_\partial^k$, then for some integer k , $[f] = [c][c'] [c' \circ c'']^k = [c][c'' \circ (c' \circ c'')]^{k-1}$. Being conjugate to a real structure, $c'' \circ (c' \circ c'')^{k-1}$ is a real structure. Thus, $[f]$ is real. \square

With similar hands-on approach as the one above, we prove the following further result.

Proposition 19. *Up to isomorphism preserving the identification $T^2 \rightarrow F_d$, there are exactly two Lefschetz fibrations with exactly two singular fibers filling the canonical open book on \check{M} ; moreover, neither of the fibrations admit a real structure.*

Proof: Let u', v' be two simple curves on T^2 such that $t_{v'} \circ t_{u'} = f$. We have $[f] = [t_v][t_u] = \begin{bmatrix} -39 & 25 \\ -25 & 16 \end{bmatrix}$. Suppose v' is the curve defined by $\alpha a + \beta b$, then a simple calculation gives $[t_{v'}] = \begin{bmatrix} 1 - \alpha\beta & \alpha^2 \\ -\beta^2 & 1 + \alpha\beta \end{bmatrix}$. All Dehn twists are represented by matrices whose traces have absolute value equal to 2. Therefore, from $[t_{u'}] = [t_{v'}^{-1} f]$, we get the identity $25\alpha^2 + 25\beta^2 - 55\alpha\beta = 25$. This quadratic Diophantine equation has solutions $(\pm 1, 0)$, $(0, \pm 1)$ and (the transpose of) these vectors multiplied from left by the powers of the matrix $S = \begin{bmatrix} -3 & 5 \\ -5 & 8 \end{bmatrix}$ (see e.g the step-by-step computation of the online application [8]). Note that u and v are in the solution set. Note also that u', v' , the curves with class b and $5a + 8b$ respectively, are in the solution set too. The pairs (t_u, t_v) and $(t_{u'}, t_{v'})$ are not Hurwitz equivalent. Indeed in that case there would be an invertible matrix K with determinant 1, such that either

$$K^{-1}[t_u^{-1}][t_v][t_u]K = [t_{u'}] \text{ and } K^{-1}[t_u]K = [t_{v'}]$$

or

$$K^{-1}[t_v^{-1}][t_u][t_v]K = [t_{u'}] \text{ and } K^{-1}[t_v]K = [t_{v'}].$$

However, by straightforward calculation one can show that there exists no such K .

Note also that the matrices S and $[f]$ commute; hence for any pair (x, y) such that $f = t_y \circ t_x$, the factorization $(S^k x, S^k y)$ is Hurwitz equivalent (with a fixed identification) to the factorization (x, y) for every $k \in \mathbb{Z}$.

As a consequence, we have two Hurwitz equivalence classes given by the pairs (t_u, t_v) and $(t_{u'}, t_{v'})$. To finish, note that the number of intersection points of u' and v' is 5, so the proof of Proposition 15 applies to (u', v') to show that the fibrations defined by the decomposition $(t_{u'}, t_{v'})$ is not real. \square

Theorem 20. *The canonical open book on \check{M} cannot be filled by any real Lefschetz fibration with the same fiber topology and with arbitrary number of singular fibers.*

Proof: We will show that \check{M} cannot be filled by a (real or non-real) Lefschetz fibration with the number of singular fibers different from 2. Recall that

$$\text{Map}(T^2 \setminus \text{int} D^2, \partial) = \{[t_a], [t_b] : [t_a][t_b][t_a] = [t_b][t_a][t_b]\}.$$

Therefore, the homomorphism

$$\begin{aligned} \deg : \text{Map}(T^2 \setminus \text{int} D^2, \partial) &\rightarrow \mathbb{Z} \\ t_a, t_b &\mapsto 1 \end{aligned}$$

is well-defined; hence the number of Dehn twists that may constitute a given element h equals $\deg(h)$. In our case, $\deg(f) = 2$, so it cannot admit a factorization as a product of an arbitrary number of positive Dehn twists other than 2. Moreover, all possible factorizations into a product of two Dehn twists are shown above to be non-real. \square

Remark 3. Elements admitting a factorization into a product of two Dehn twists will be studied in [5]. There the classification of such factorizations up to Hurwitz equivalence are presented and their relation to real structures are also elaborated.

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BOĞAZIÇI ÜNİVERSİTESİ, DEPARTMENT OF MATHEMATICS, TR-34342 BEBEK, İSTANBUL, TURKEY
E-mail address: ferit.ozturk@boun.edu.tr

INSTITUT CAMILLE JORDAN, UNIVERSITÉ LYON I, 43, BOULEVARD DU 11 NOVEMBRE 1918 69622
 VILLEURBANNE CEDEX, FRANCE
E-mail address: salepci@math.univ-lyon1.fr